A USER'S GUIDE TO

THE INFLATION GENERATOR

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The level and change of the rate of inflation is an important assumption in many actuarial models. The rate of inflation can affect the level of future claims, the demand for new policies, as well as affect the level of interest rates or returns on assets. This paper documents the inflation generator that has been developed for the sponsoring organizations and is available for a variety of actuarial applications. The model simulates the general inflation rate and the user may wish to translate the general inflation rate to other rates such as line of business inflation.

The paper is organized as follows. The first section describes a general process for inflation that is used in the spreadsheet model. This section focuses on explaining the effects of changes in the model's parameters on inflation projections. The second section provides more details about the inflation generator when using monthly time steps in the simulation. The third section discusses regime switching where the inflation process is modified based on potential structural changes in the economic environment. Finally, we illustrate the inflation generator with actual screenshots from the spreadsheet model.

1. MEAN REVERTING INFLATION UNDER ANNUAL TIME STEPS

A mean reverting process

There are many possible formulations of an inflation model. The inflation generator uses an Ornstein-Uhlenbeck process which is based on *changes* in the inflation rate. The continuous time Ornstein-Uhlenbeck process is:

$$dq_t = k(\theta - q_t)dt + \sigma dB_t \tag{1}$$

This process is a mean reverting process since the level of inflation tends toward some average level denoted by θ. The first term on the right hand side of equation (1) is called the drift and it indicates the expected change in the level of inflation over the next interval of time. The second term is called the diffusion and it represents the volatility, or stochastic component of future inflation.

To understand the mean reverting process, consider the case where the current level of inflation (q_t) is below its mean, so that ($q_t < \theta$). In this case, the drift is positive, which indicates that inflation is expected to increase next period. How quickly the rate of inflation reverts to its mean is determined by the speed parameter k. As the drift pulls inflation toward θ, there is uncertainty in the environment which is represented by changes in a Brownian motion (dB_t). The relative size of the parameters k and σ affect the amount of randomness of the inflation process. If σ is large, the uncertainty exhibited by the Brownian motion is magnified and any reversion toward the mean inflation rate may be overshadowed by the diffusion process. However, if k increases (relative to σ), then mean reversion dominates the movement of inflation.

Though continuous time processes are useful for deriving analytical results such as the average future inflation rate, actuaries often use discrete time simulations when performing cash flow testing or

dynamic financial analysis (DFA) where cash flows are related to inflation. Using annual time steps, the discrete time equivalent of equation (1) is an autoregressive time series model:

$$q_{t+1} = q_t + dq_t = [(1-k)q_t + k\theta] + \sigma\varepsilon_t \quad (2)$$

The first term on the right hand side of equation (2) (in square brackets) shows that the expected future inflation is an average of two values: the current level of inflation (q_t) and the mean reversion level θ. The parameter k determines the relative weight attached to current environment and a long-term average. If mean reversion speed is high, then recent history is not weighted heavily and inflation quickly reverts to θ. The second term in (2) adds random shocks based on a draw from a normalized distribution (ε_t), which is scaled by a constant volatility parameter σ which adjusts the amount of uncertainty in the inflation process.

Numerical examples – Annual time steps

A few examples will help illustrate how equation (2) works and the implications of each of the model's parameters on projected inflation levels. (NOTE: In the examples that follow, the parameters are chosen for illustration purposes only to isolate the effects of each parameter.)

In the first example, the mean reversion speed is set at a high level which weights the long-term average heavily when projecting future inflation. (It might be noted that with a discrete time process with annual time steps, choosing a mean reversion speed greater than 1.0 leads to excessive fluctuations in inflation by creating a "reflective" process. If $k > 1.0$ and the current level of inflation is below its mean, next year's inflation is expected to *exceed* the mean.)

We project 10,000 paths for the level of inflation under several alternate parameters. The parameters for the first simulation (Example A) are shown in Table 1.

TABLE 1

Sample Parameters for Simulation Example A (Annual Time Steps)

Variable	***Value***
k	1.0
θ	3.00%
σ	4.00%
Initial inflation	1.00%

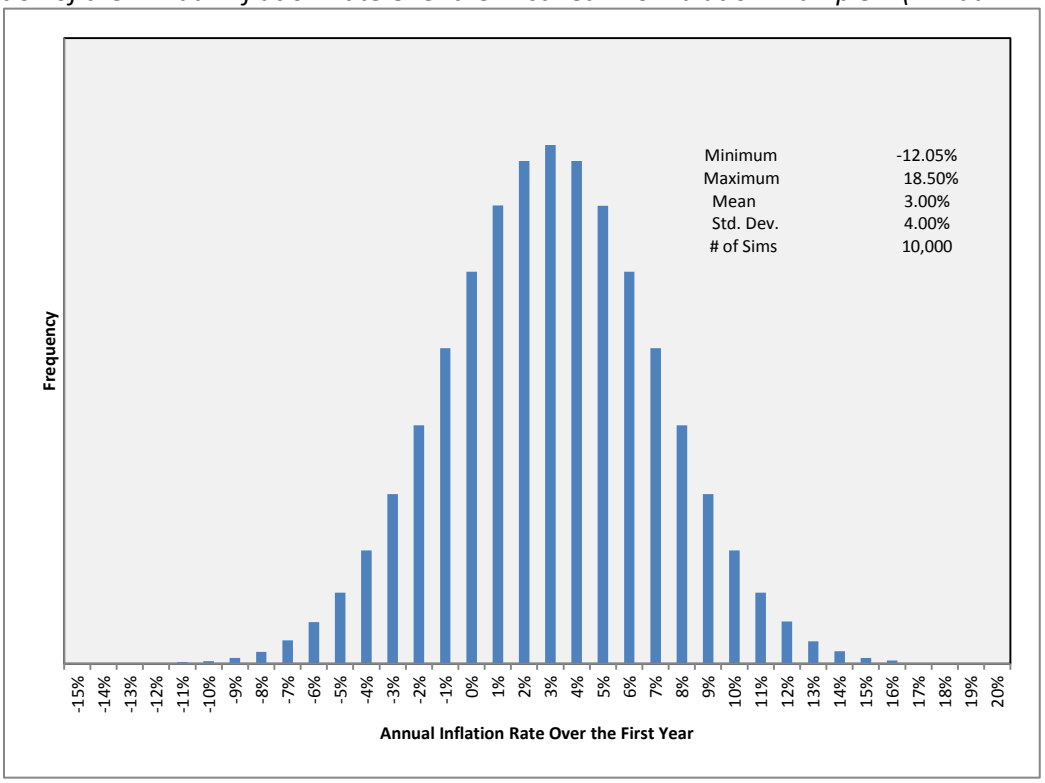
The probability distribution of projected inflation from Simulation Example A is shown in Figure 2 below. To be clear, the distribution illustrates one particular inflation rate from the model, the one-year inflation rate over the first projection year (see Figure 1).

FIGURE 1

Illustration of Projection Period – Annual Inflation Rate in One Year

FIGURE 2

Distribution of the Annual Inflation Rate Over the First Year – Simulation Example A (Annual Time Steps)

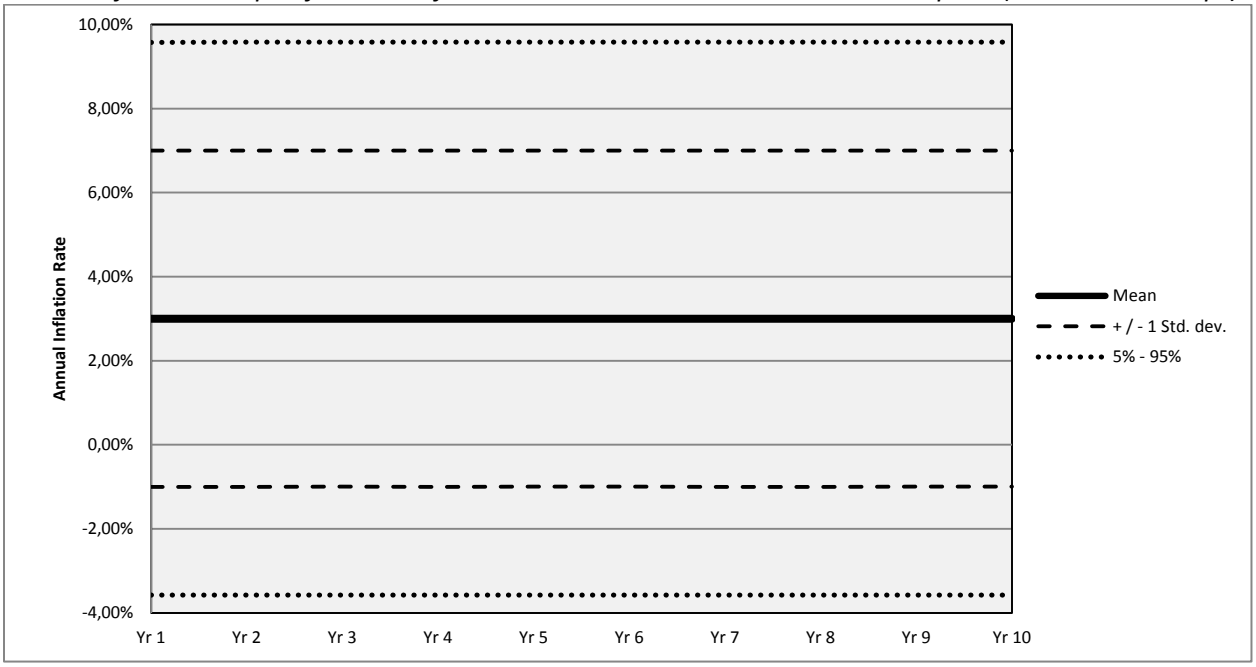

Minimum -12.05%
Maximum 18.50%
Mean 3.00%
Std. Dev. 4.00%
# of Sims 10,000
Frequency
-15% -14% -13% -12% -11% -10% -9% -8% -7% -6% -5% -4% -3% -2% -1% 0% 1% 2% 3% 4% 5% 6% 7% 8% 9% 10% 11% 12% 13% 14% 15% 16% 17% 18% 19% 20%
Annual Inflation Rate Over the First Year

With Simulation Example A, k = 1.0 so that no weight is attached to the current level of inflation (see equation (2)). Because the mean reversion speed is so high, the inflation process has no memory and the expected rate of inflation coincides with the mean reversion level of 3%. In fact, the distribution of future inflation is a normal distribution with mean equal to θ = 3% and σ = 4%. Changing the volatility parameter (σ) simply affects the uncertainty surrounding expected inflation so that increasing the volatility in the above example simply leads to a wider normal distribution, but the mean is unaffected.

Figure 3 below depicts the distribution of annual inflation rates over time, from year one to year ten. Figure 3 shows the mean annual inflation rate over time, as well as two measures of dispersion including the standard deviation and the tails of the distribution (the 5^{th} and 95^{th} percentiles). This type of graph is often called a "funnel of doubt" graph and was introduced by Redington (1952). Given the limited memory feature of the process when k = 1.0, all future inflation rates are Normal(θ, σ).

FIGURE 3

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example A (Annual Time Steps)



Simulation Example A generates a simple distribution for future inflation, but it is likely that the expected path of inflation does place weight on history. While central bankers may employ various tools of monetary policy to keep inflation around an established target level of inflation, these tools take time to propagate through the economy. In fact, excessive use of monetary policy tools may lead to increased economic volatility. Therefore, it is reasonable to assume that the expected future rate of inflation adjusts more slowly over time, with some weight attached to current inflation levels. To incorporate the more modest change in inflation over time, we lower the level of mean reversion speed for Simulation Example B. In order to compare the results to the previous projections, all of the remaining parameters are left unchanged.

TABLE 2

Sample Parameters for Simulation Example B (Annual Time Steps)

Variable	***Value***
k	0.5
θ	3.00%
σ	4.00%
Initial inflation	1.00%

Using these parameters, each year the rate of inflation is expected to move only half of the distance from its current value of 1.0% toward the reversion level of 3.0%. For Simulation Example B, annual inflation over the next year still has a normal distribution since the source of randomness in equation (2) is ε, a standard normal variate. Figure 4 illustrates the distribution of projected inflation next year, which is normal with a mean of 2% and a standard deviation equal to the volatility parameter of 4%.

FIGURE 4

Distribution of the Annual Inflation Rate over the First Year – Simulation Example B (Annual Time Steps)

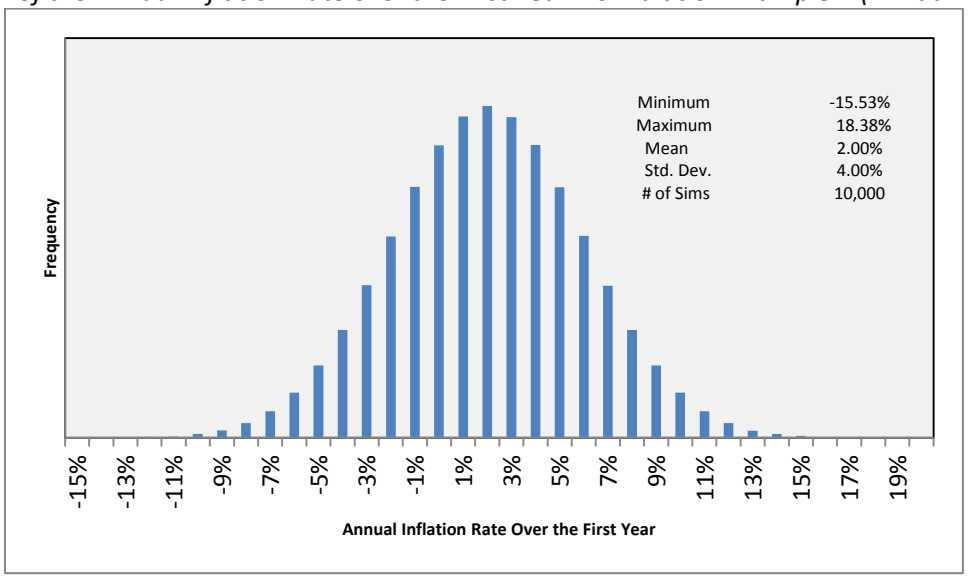
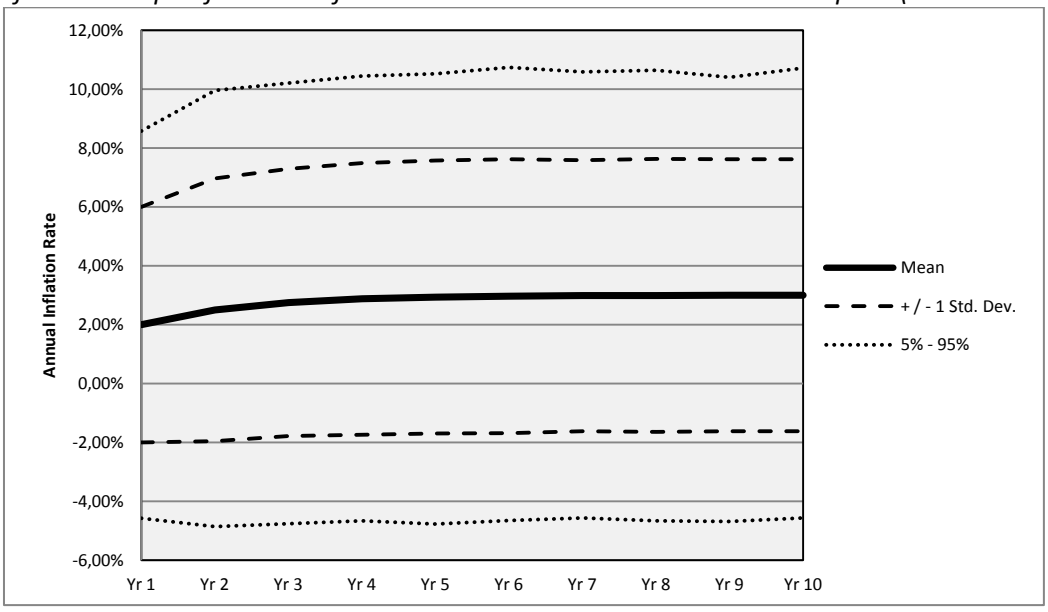

Minimum -15.53%
Maximum 18.38%
Mean 2.00%
Std. Dev. 4.00%
# of Sims 10,000
Frequency
-15% -13% -11% -9% -7% -5% -3% -1% 1% 3% 5% 7% 9% 11% 13% 15% 17% 19%
Annual Inflation Rate Over the First Year

Figure 5 is the funnel of doubt graph for Simulation Example B and the only apparent difference is the slow increase in the mean level inflation over the next 10 years since each year the expected level of inflation moves half of the way from its current level to the mean level of 3%. Upon further inspection of Figure 5, it becomes clear that the volatility also grows slightly as we project further into the future.

FIGURE 5

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example B (Annual Time Steps)



Let us look more closely at the one-year inflation rate in the second projection year (depicted in Figure 6 below).

FIGURE 6

Illustration of Projection Period – Annual Inflation Rate in Two Years

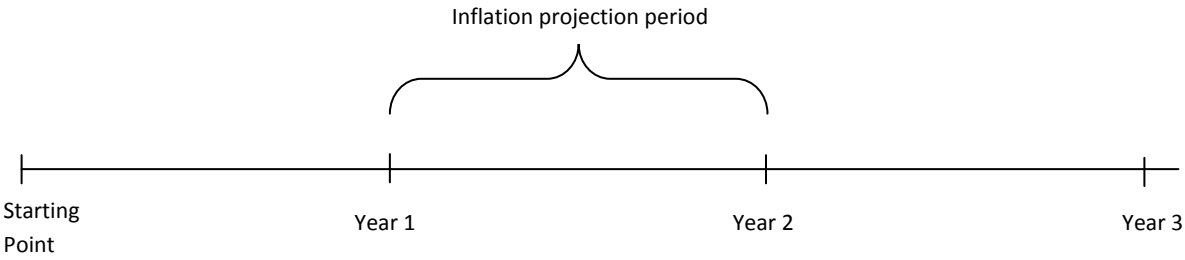
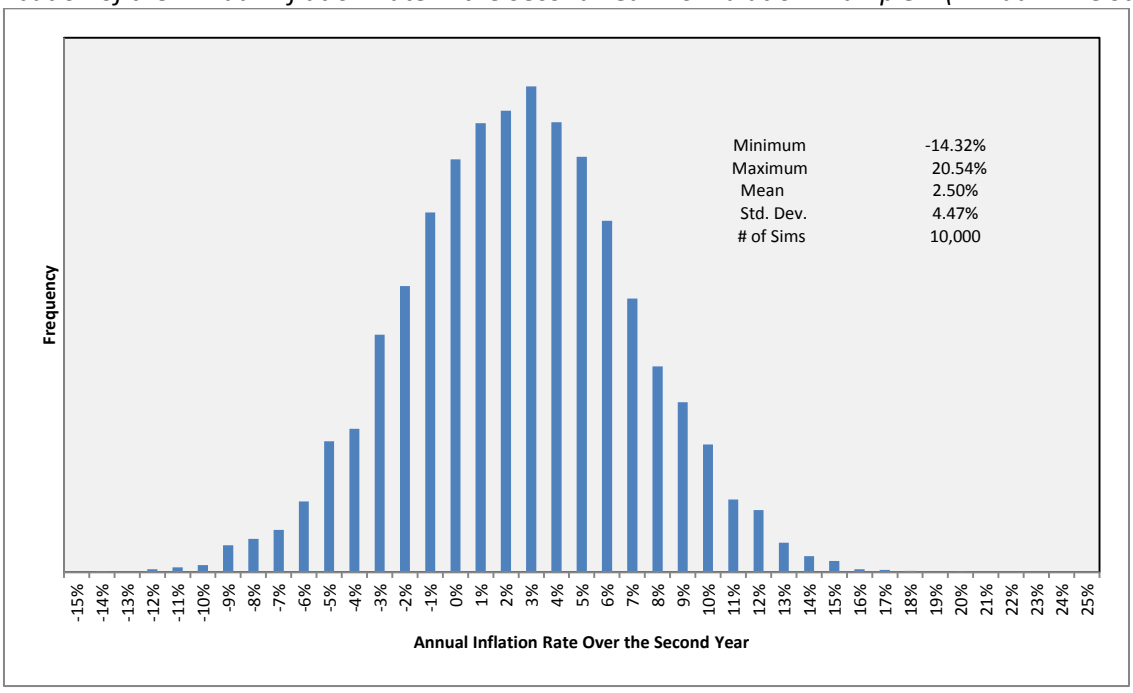


Figure 7 below shows the distribution of the projected annual inflation rate in the second projection year. Note that each year the rate of inflation is expected to move half of the distance toward its mean level of 3%. After one year, expected inflation moves from the initial level of 1% to 2% (as shown above), and in the second year (expected) inflation again moves half its distance toward 3%, from 2% to 2.5%. However, in the second projection year, the standard deviation has increased from 4% to 4.47%.

FIGURE 7

Distribution of the Annual Inflation Rate in the Second Year – Simulation Example B (Annual Time Steps)



When making longer-term inflation projections in Simulation Example A, there was no difference between the distribution of projected inflation in the first year vs. any other future year (e.g., the 10th projection year). With high mean reversion speed (k = 1.0) , the inflation process "lost" its memory quickly since there was no weight applied to history; any random noise created by the economy in the first projection year was completely forgotten when projecting inflation in future years. As a result, the standard deviation of the distribution of future inflation rates was completely driven by the contemporaneous uncertainty inherent in the economy.

But when the speed of reversion is slower, the inflation process remembers the uncertainty of the past so that outliers in early projection years carry over to future years. In Simulation Example B, future

inflation is based on the existing level of inflation and the assumed long-run mean. And the volatility of future inflation increases because it reflects the uncertainty inherent in the projection period is weighted with the volatility exhibited in previous periods. To give an economic interpretation, policymakers must deal with the imprecise data of the current period, as well as consider the significance of noisy past data.

This growth in volatility is not unbounded for longer projection periods. As long as some weight is attached to a target mean reversion level, the increase in volatility is limited since historical levels of inflation become less relevant as the projection period increases. Miller and Wichern (1977) show how the volatility of an autoregressive process increases with the projection period. In particular, they calculate the volatility limit for long projection periods. As the projection period (t) increases, Miller and Wichern (1977) find the limit of volatility for an autoregressive process with annual time steps (such as equation (2)) is:

$$Volatility\ q_t \rightarrow \frac{\sigma}{\sqrt{1-(1-k)^2}} \tag{3}$$

Table 3 illustrates the growth of volatility by projection period and the effects of varying the speed of mean reversion of the inflation process. When the mean reversion speed is high (Simulation Example A), the process has memory loss and as the projection period grows, volatility is tied to the current environment. However, when k is low, the inflation process is less tethered to an average inflation level and uncertainty grows.

TABLE 3

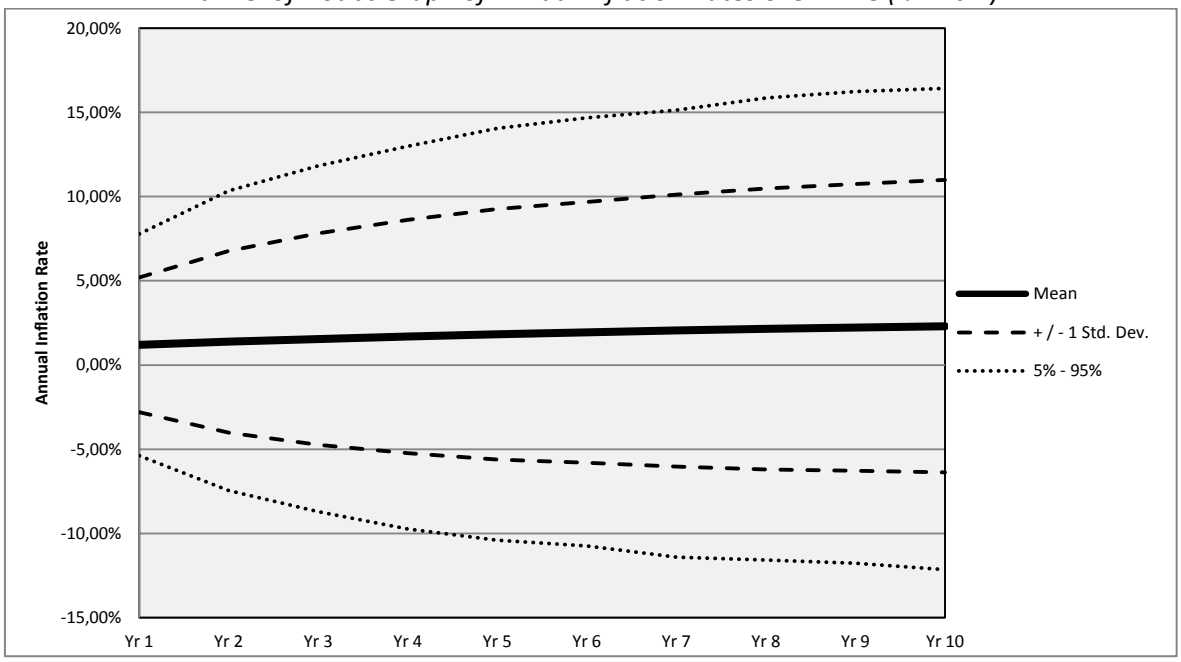
Comparison of Volatility by Projection Period

Projection Year	***Std. Dev. of Inflation*** $(k = 1.0)$	***Std. Dev. of Inflation*** $(k = 0.5)$	***Std. Dev. of Inflation*** $(k = 0.1)$
1	4.00%	4.00%	4.00%
2	4.00	4.47	5.38
3	4.00	4.58	6.28
4	4.00	4.61	6.93
5	4.00	4.62	7.41
6	4.00	4.62	7.77
7	4.00	4.62	8.06
8	4.00	4.62	8.28
9	4.00	4.62	8.46
10	4.00	4.62	8.60
⋮	⋮	⋮	⋮
∞	4.00	4.62	9.18

Figure 8 below translates this growth in volatility to a funnel of doubt graph for a simulation with k = 0.1. Comparing Figure 8 to earlier funnel of doubt graphs (Figures 3 and 5), we can see how uncertainty changes with the mean reversion speed.

FIGURE 8

Funnel of Doubt Graph of Annual Inflation Rates over Time ($k = 0.1$)



Changes in mean and volatility parameters

Much of the discussion until now has centered on the speed of mean reversion parameter k. Simulation Example C illustrates the effects of changes in the mean level of inflation and the volatility, which have more straightforward interpretations. Increasing θ leads to higher expected inflation projections during all projection years while increasing σ leads higher levels of uncertainty of future inflation. The parameters for Simulation Example C are shown in Table 4 below:

TABLE 4

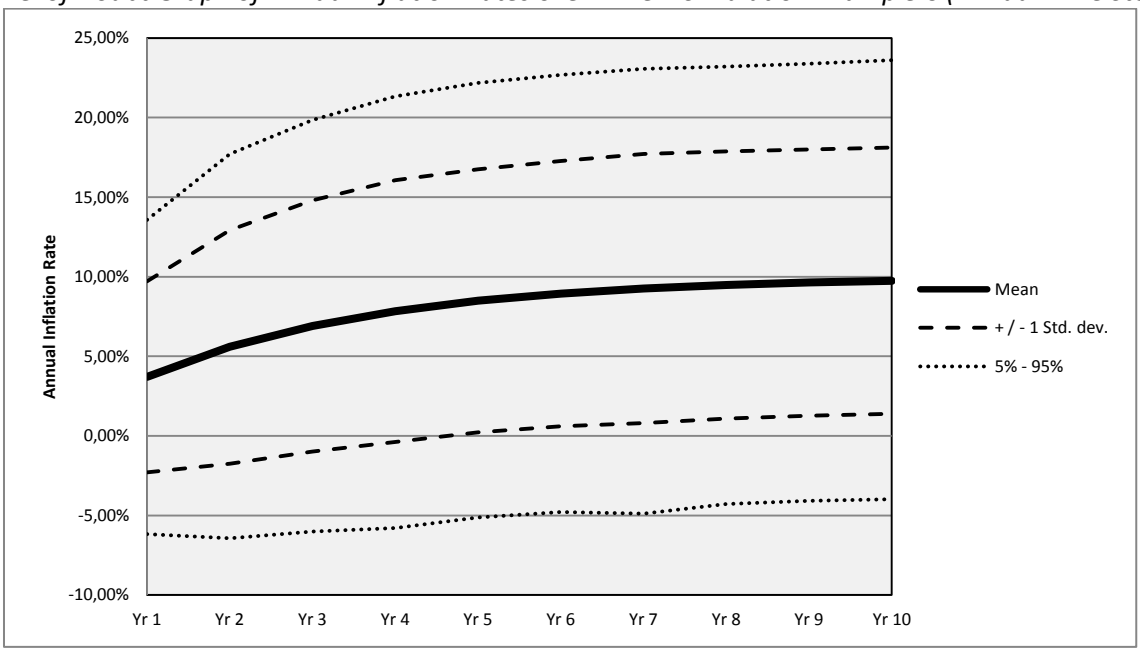
Sample Parameters for Simulation Example C

Variable	***Value***
k	0.3
θ	10.00%
σ	6.00%
Initial inflation	1.00%

Figure 9 is the funnel of doubt graph showing the distribution of annual inflation rates for years one through 10.

FIGURE 9

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example C (Annual Time Steps)



As expected, the average inflation level slowly increases toward 10% over the projection period. If the modeler wants to have expected inflation move more quickly to 10%, he or she can increase the mean reversion speed. In addition, the standard deviation of the distribution starts out at 6% and increases slightly over the remaining projection period. Based on the discussion of the two previous example simulations, the annual inflation rate in the first projection year has a normal distribution with mean of 3.7% and a standard deviation of 6.00% (σ). The mean is 3.7% since it is 30% of the way from the initial rate of 1% toward the long-term mean level of 10%. In the 10th projection year, the mean simulated level of inflation is 9.75% and the standard deviation is 8.43%.

2. MEAN REVERSION WITH MONTHLY TIME STEPS

Changing the time step

The discussion above uses examples with annual time steps to help illustrate the basic operation of an autoregressive model. However, the inflation generator uses monthly time steps to project inflation. Extending the discrete autoregressive model (2) to shorter time steps yields:

$$q_{t+\Delta t} = \{[1 - k(\Delta t)]q_t + k(\Delta t)\theta\} + \left(\sigma\sqrt{\Delta t}\right)\varepsilon_t \qquad (4)$$

With annual time steps, it was easier to see the impact of the speed of reversion parameter (k) since it could be directly interpreted as the weighting applied to the mean reversion level (see equation (2)). Thus, when k = 1.0 and there are annual time steps, the process has no memory since there is no weight attached to previous inflation values. But when the time step is reduced, things become a bit more complicated.

Consider how the expected inflation rate is affected if we iterate (4) when k = 1.0 using semiannual time steps (Δt = 0.5). The expected inflation becomes:

$$\begin{gathered} E(q_{0.5}) = \{[1 - k(\Delta t)]q_0 + k(\Delta t)\theta\} \\ = 0.5 \times 1\% + 0.5 \times 3\% = 2.0\% \\ \\ E(q_{1.0}) = \{[1 - k(\Delta t)]q_{0.5} + k(\Delta t)\theta\} \\ = 0.5 \times 2\% + 0.5 \times 3\% = 2.5\% \end{gathered} \tag{5}$$

Since k = 1.0, inflation does move toward the mean reversion level of 3.0%. But since the time steps are semiannual, inflation moves only half of that distance during *each* time step. After two semiannual time steps, the expected inflation rate is 2.5%. Compare this to the annual time step process with the same parameters, where expected inflation would have moved fully to its mean reversion level. To project future inflation, we iterate equation (4):

$$E(q_{t+s\Delta t}) = [1 - k(\Delta t)]^s \times q_t + \left[1 - \left(1 - k(\Delta t)\right)^s\right] \times \theta \quad \text{for } s = 1, 2, \ldots \tag{6}$$

While future inflation is still a weighted average of past values and the mean reversion level, the *annual* mean reversion speed has less direct interpretation and subsequent inflation projections are affected.

The inflation process (q_t) vs. observed annualized inflation ($\varphi_{t,t+s}$)

Equation (6) determines the level of the expected inflation process (q) at some future time period. When Δt < 1.0, we need to make a distinction between expressing the value of the contemporaneous inflation process vs. expressing annualized inflation. The inflation generator uses monthly time steps, so the expected value of the inflation process (q) from equation (6) only affects price levels for *one month*. However, when discussing inflation, it is common to think of the annual change in prices from one year to the next. This view of inflation is an historical, cumulative effect of the inflation process over 12 consecutive months.

To illustrate the difference, first define $\varphi_{t,t+s}$ as the observed and annualized inflation rate over an s-year period, from time t to time $t + s$, where s > 0. A common way to measure $\varphi_{t,t+s}$ is by determining the percentage change in the level of prices as measured by the consumer price index (CPI):

$$\left(1 + \varphi_{t,t+s}\right)^s = \frac{CPI_s}{CPI_t}$$

Now suppose we wanted to use the inflation generator to estimate the rate of inflation during the first projection year. To calculate this value (denoted $\varphi_{0,1}$), we need to compound the simulated value of the inflation process each month over the next 12 months. Let $q_{1/12}$, $q_{2/12}$, …, $q_{12/12}$, be the sampled inflation process from (4) using monthly time steps. Then,

$$\left(1 + q_{1/12}\right)^{1/12}\left(1 + q_{2/12}\right)^{1/12} \ldots \left(1 + q_{12/12}\right)^{1/12} = 1 + \varphi_{0,1} \tag{7}$$

In the simulations performed in the prior section using annual time steps, there was no need to distinguish the inflation process and the reported rate of inflation since they were identical. But

equation (7) indicates that observed annual inflation is based on projections of the inflation process at different points in time.

For completeness, as the number of time steps in a year increases, the limit is the continuous time model in equation (1) and the annualized inflation is:

$$1 + \varphi_{0,1} = exp\left(\int_0^1 q_t dt\right)$$

Monthly time step simulations

In this section, we run the same simulations that were illustrated in the first section using the same parameters. However, as in the inflation generator, in this section we use monthly time steps instead of annual time steps.

TABLE 5

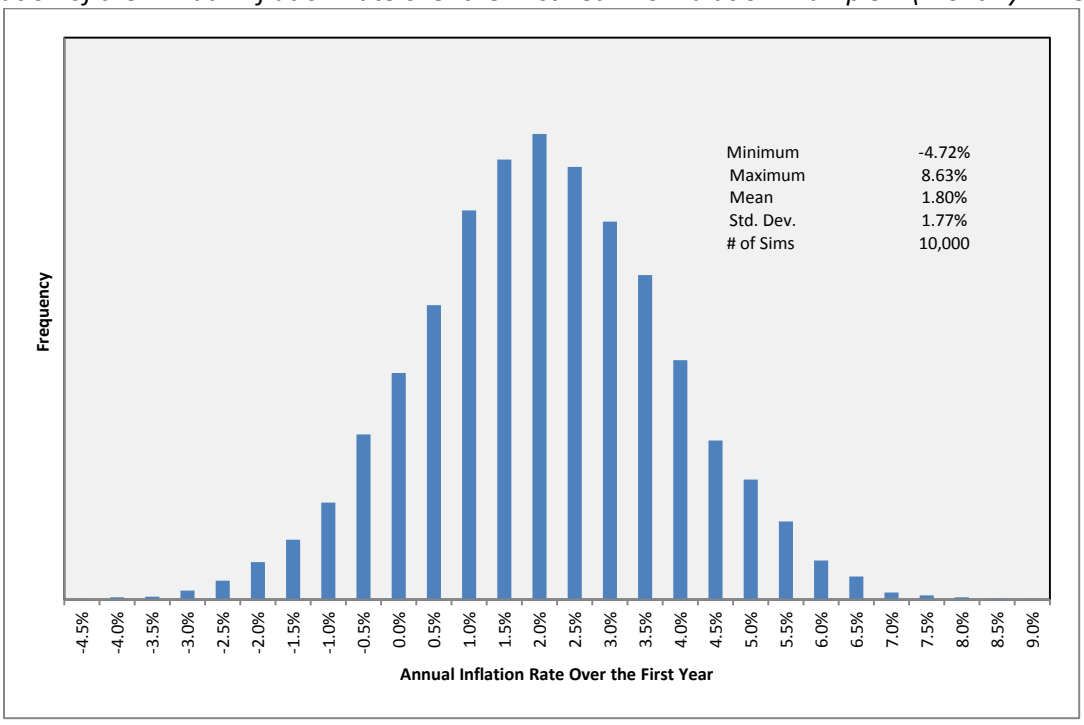
Sample Parameters for Simulation Example A (Monthly Time Steps)

Variable	***Value***
k	1.0
θ	3.00%
σ	4.00%
Initial inflation	1.00%

Figure 10 illustrates the distribution of the annual inflation rate resulting from 10,000 projections paths of the monthly inflation process. (For comparative purposes, the distribution of first year inflation using annual time steps was shown in Figure 2.)

FIGURE 10

Distribution of the Annual Inflation Rate over the First Year – Simulation Example A (Monthly Time Steps)


Minimum -4.72%
Maximum 8.63%
Mean 1.80%
Std. Dev. 1.77%
# of Sims 10,000
Frequency
-4.5% -4.0% -3.5% -3.0% -2.5% -2.0% -1.5% -1.0% -0.5% 0.0% 0.5% 1.0% 1.5% 2.0% 2.5% 3.0% 3.5% 4.0% 4.5% 5.0% 5.5% 6.0% 6.5% 7.0% 7.5% 8.0% 8.5% 9.0%
Annual Inflation Rate Over the First Year

Note that the mean value of annualized inflation is 1.80%, which is significantly lower than the case with annual time steps (3.00%) for two reasons. First, as illustrated in the example in equation (5), when moving to shorter time steps the interpretation of the mean reversion speed parameter k is changed. One way that is helpful to interpret the speed of mean reversion of the inflation process (q) toward its long-term average is its "half-life," or the time it takes to move one-half of the way to the mean reversion level.

$$\text{Half-life (\# of periods)} = \frac{\ln(0.5)}{\ln[1-k(\Delta t)]} \tag{8}$$

When k = 1.0 with annual time steps, inflation was expected to move to the reversion level of 3% next period. However, with monthly time steps (Δt=1/12), equation (8) shows that it takes approximately 8 months for the inflation process to make it half of the distance toward 3%. In fact, Table 6 shows the expected value of the inflation process (q_t) from Simulation Example A for each month during the first year.

TABLE 6

Expected Path of Inflation Process – Simulation Example A (Monthly Time Steps)

Month	***Expected Value of Inflation Process (q)***
Initial	1.00%
1	1.17%
2	1.32%
3	1.46%
4	1.59%
5	1.71%
6	1.81%
7	1.91%
8	2.00%
9	2.09%
10	2.16%
11	2.23%
12	2.30%

The second difference in expected inflation when using monthly time steps is that Figure 10 shows the distribution of $\varphi_{0,1}$ which is the cumulative change in prices over the first year. Equation (7) shows that the construction of the annualized inflation rate is developed from the projections of the monthly inflation process. Given that the inflation process starts at 1% and is expected to move to monthly toward 3% slowly over time according to Table 6, it turns out the average inflation rate over this first year is 1.80%.

The funnel of doubt graph for Simulation Example A under monthly time steps is shown in Figure 11. The rate of annual inflation slowly increases toward 3% each year (as expected).

FIGURE 11

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example A (Monthly Time Steps)

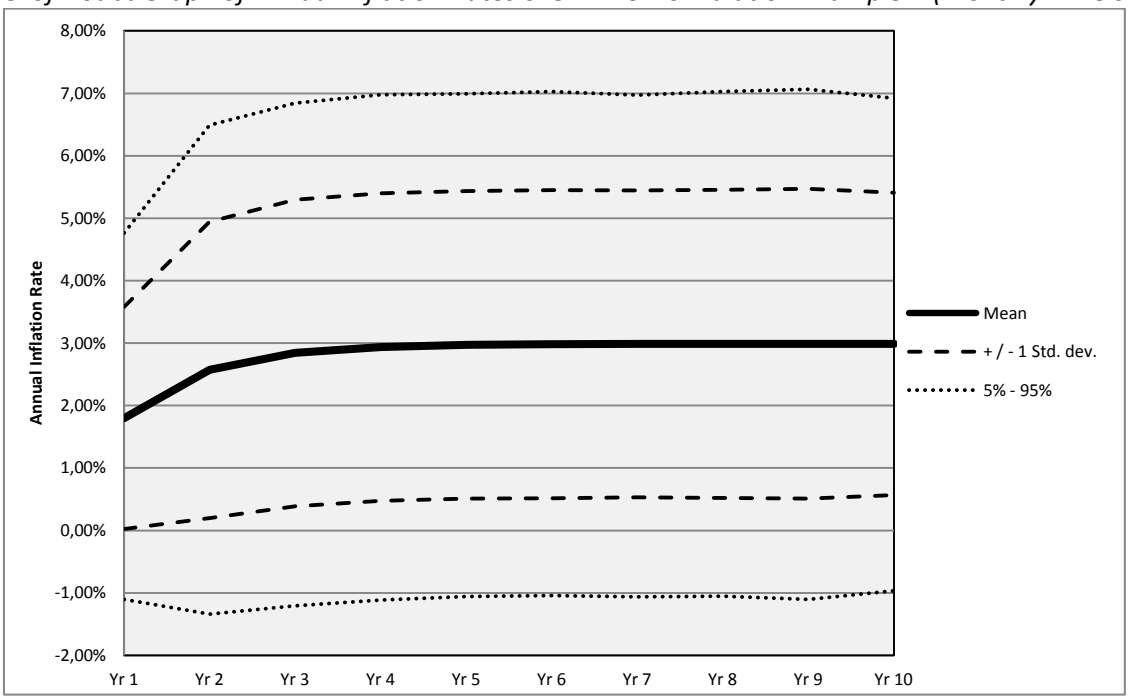


Figure 10 also states the standard deviation of the first year inflation rate as 1.77%, again lower than the 4.00% when using annual time steps. With annual time steps and k = 1.0, uncertainty in future inflation was driven completely by the volatility parameter σ since there was no distinction between q and φ. Two forces influence the volatility when changing to monthly time steps.

First, the volatility of the monthly inflation process is now scaled by the time step ($\sigma\sqrt{\Delta t}$) and while Figure 11 shows that the volatility increases over time, it is less than the volatility parameter of the inflation process (σ = 4%). As forecast period is extended, equation (3) illustrates how the volatility of q_t grows and if we restate equation (3) with monthly time steps:

$$Volatility\ q_t \rightarrow \frac{\sigma\sqrt{\Delta t}}{\sqrt{1-[1-k(\Delta t)]^2}} \tag{9}$$

In Simulation Example A, this limit of volatility is 2.89%. Thus, for very long term projections of q_t, the standard deviation is approximately 2.89%, less than the volatility with annual time steps.

As explained earlier, the relative sizes of the mean reversion speed and the volatility parameters work together to determine the amount of randomness in inflation. In Simulation Example A, the high level of mean reversion dampens the effects of uncertainty over time. If there is a month where the random shock happens to be an outlier, the high level of mean reversion quickly brings inflation back toward θ. This dramatically reduces the amount of uncertainty in inflation over any period of time. However, as the size of mean reversion falls (relative to σ), outliers can compound and lead to more uncertainty when predicting inflation. Table 7 shows the effects of reducing mean reversion speed on long-term inflation projections as determined by equation (9):

TABLE 7

Limit of Volatility of Inflation Process

k	***Volatility limit (eq. (9))***
1.0	2.89%
0.5	4.04%
0.3	5.20%
0.2	6.35%
0.1	8.96%

The second factor affecting the volatility of annual inflation depicted in Figures 10 and 11 is based equation (7), which is the product of the monthly inflation process. The standard deviation of annualized inflation is based on the volatility of a product of the inflation process each month. Goodman (1962) provides a general analytic solution to the variance of the product of random variables. The numerical solution for the standard deviation of $\varphi_{0,1}$ is 1.77% as shown in Figure 10. In the 10^{th} projection year, the standard deviation is 2.45%.

Simulation Example B lowers the mean reversion speed.

TABLE 8

Sample Parameters for Simulation Example B (Monthly Time Steps)

Variable	***Value***
k	0.5
θ	3.00%
σ	4.00%
Initial inflation	1.00%

Figure 12 illustrates the distribution of the projected inflation in the first year and Figure 13 provides the funnel of doubt graph for Simulation Example B using monthly time steps.

FIGURE 12

Distribution of the Annual Inflation Rate over the First Year – Simulation Example B (Monthly Time Steps)

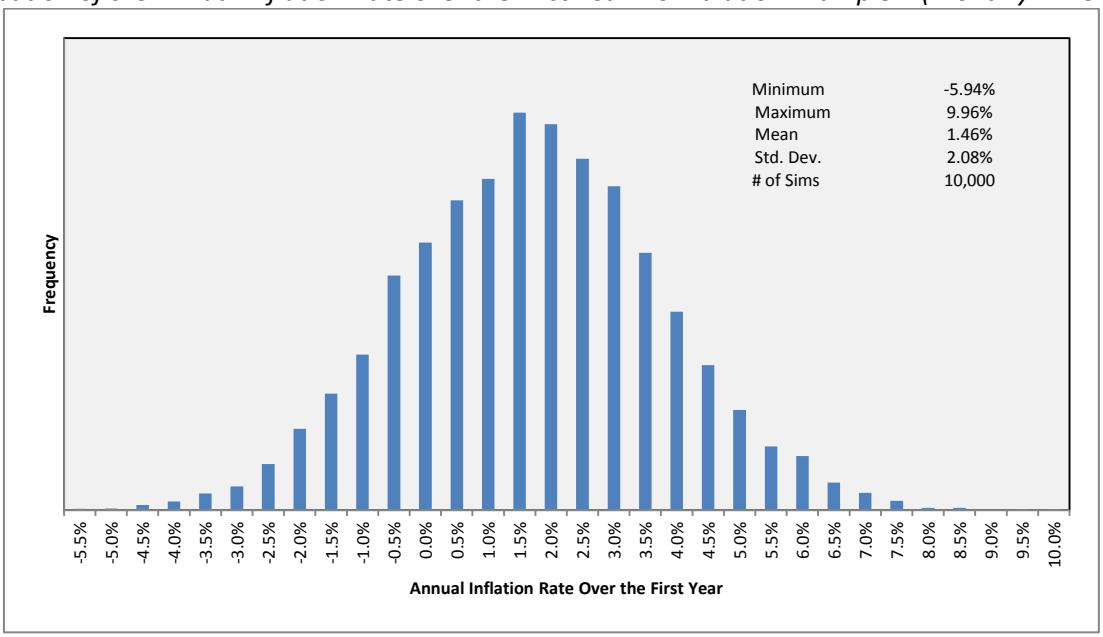
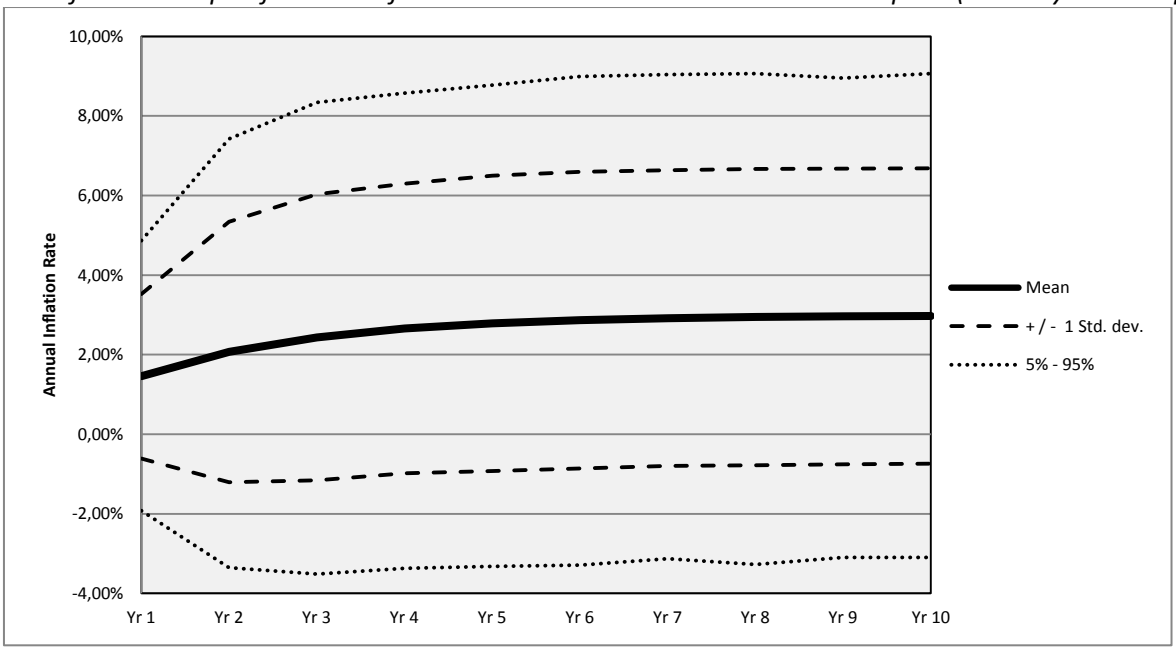


FIGURE 13

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example B (Monthly Time Steps)



Note that when the reversion speed is lowered, it takes more time for inflation to move toward θ and the volatility of inflation is higher.

The results of Simulation Example C are quickly shown in Figure 14.

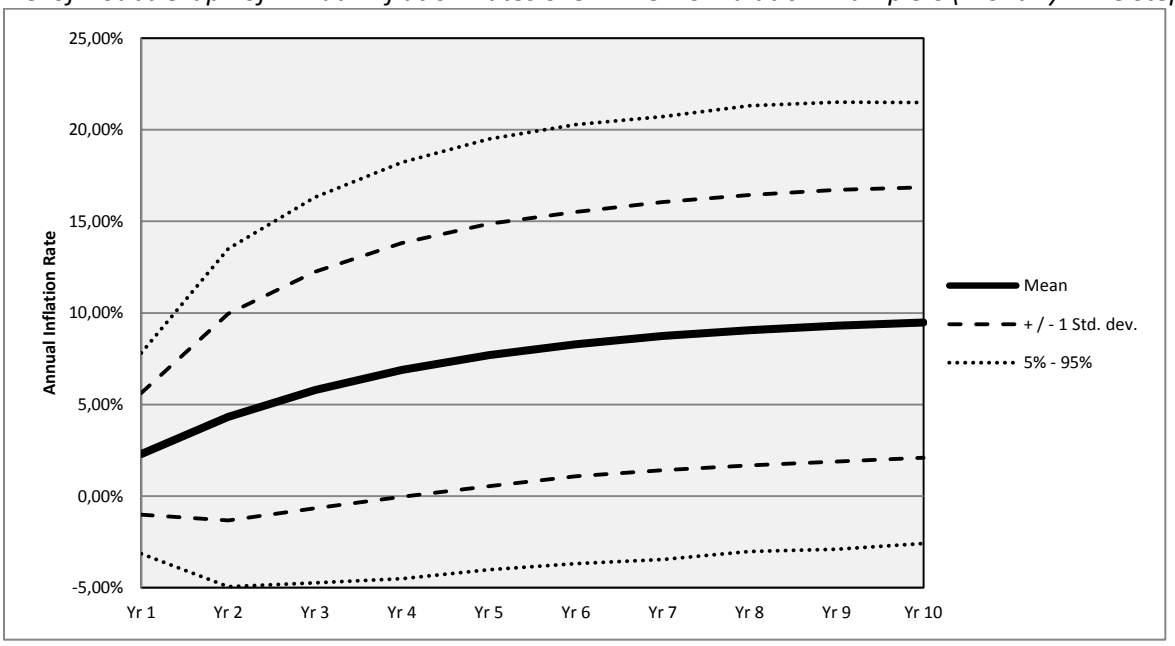
TABLE 9

Sample Parameters for Simulation Example C (Monthly Time Steps)

Variable	***Value***
k	0.3
θ	10.00%
σ	6.00%
Initial inflation	1.00%

FIGURE 14

Funnel of Doubt Graph of Annual Inflation Rates over Time – Simulation Example C (Monthly Time Steps)



3. REGIME SWITCHING

When developing models for economic and financial variables, history is often used as a metric of model performance. Often there exists historical subperiods that occur during structural transitions in the economy or during financial crises and the historical fit of models may be lacking.

Consider term structure models which aim to mimic interest rate movements. When measuring the relative performance of several models (such as in Chan, Karolyi, Longstaff, and Sanders (1992)), most popular models fail to capture the interest rate dynamics exhibited during the early 1980s. But the higher levels of interest rates and the increase in volatility of this period may be caused by a shift in the policies of the Federal Reserve to target money growth. In situations where there are distinct episodes where the behavior of time series appears markedly distinctive, modelers may consider a change in regime. Hamilton (1989) provides a general discussion of regime switching and Ang and Bekaert (2002) describe an application to term structure models. The economic rationale for regime switching is that at any point in time, the dynamics of a model are dictated by a particular regime. But changes in the economy may build such that the assumed process for financial variables is no longer appropriate. In

these cases, the economy is said to switch regimes. In an alternate regime, the dynamics of financial models differ from those assumed under a normal economic environment.

As another example of regime switching models, Hardy (2001) discusses a regime switching model for stock returns as a way to capture the fatter tails exhibited in historical returns. In normal economic times, equity returns may be approximately normal with a constant variance, but during times of severe economic uncertainty and recession observed returns from a normal distribution might appear statistically improbable. To handle these outliers, Hardy (2001) introduces a second regime which incorporates increased uncertainty. Typically, to keep models tractable, regimes are defined somewhat broadly so that the number of changes in a given period remains relatively low. For example, Hardy (2001) uses only two regimes and finds that extending the stock return model to three regimes yields only a marginal improvement in statistical fit for US data.

The inflation generator uses an autoregressive process to project future inflation rates. When selecting parameters for this process, users might choose values that yield a model that mimics historical inflation. But future economic conditions have the potential to move inflation in ways that have not been observed in the short history embedded in US data. In addition to "normal" inflation, we see two other possible regimes which are plausible. First, actions of the Federal Reserve targeted to stimulate the economy may create increased price pressures in the economy that could generate higher inflation than observed recently. Alternatively, if the actions of the Federal Reserve fail to stimulate significant economic activity or if the world economy continues to retrench, decreased demand may increase the potential for deflationary environments.

Given these other possible scenarios, the inflation generator uses three regimes. In each regime inflation follows an autoregressive process, but at any point in time the dynamics of the inflation process are dictated by the parameters of the prevailing regime. Changes in regime are based on transition (or switching) probabilities. Selecting transition probabilities is an important but difficult task given the inability to pinpoint specific regimes in historical data since we cannot directly observe changes in regimes but instead must imply them. Even when the level of inflation has been observed to be "low" relative to historical standards, this does not mean that the economy has entered the deflationary regime period; an alternative explanation is that the normal regime has just experienced an outlier. Single outliers are temporary, but regimes tend to persist.

The transition probabilities directly determine the expected length of time in a particular regime. Suppose the economy is currently in the "normal" regime. Denote π as the probability of staying in the normal regime next period. Therefore, the probability of switching to either to a high inflation regime or to a deflation regime is $(1-\pi)$. We can determine the probability that the current regime lasts for exactly five periods:

$$Probability\ of\ regime\ duration\ of\ five\ periods = \pi^4(1-\pi)$$

In fact, the expected (mean) number of periods for the regime duration is

$$\sum_{t=1}^{\infty} t \times \pi^{t-1} \times (1-\pi) = \frac{1}{1-\pi} \quad (10)$$

To understand how the transition probabilities affect the regime duration, consider Table 10. The first column lists the probability of remaining in a particular regime from month to month, denoted as π

above. The second column provides another interpretation by simply restating this probability another way: the probability of *switching* regimes within a year. The third column uses equation (10) to derive the mean duration in a particular regime. But since the probability distribution of a regime's duration is positively skewed, the last column also shows the median duration of a regime. Table 10 may be helpful for users choosing transition probabilities for the inflation generator.

TABLE 10

Transition Probabilities and Regime Duration

Monthly Likelihood of Remaining	***Annual Equivalent Switching***	***Mean (In years)***	***Median (In years)***
99.9%	1.2%	83.33	57.73
99.8%	2.4%	41.67	28.85
99.7%	3.5%	27.78	19.23
99.6%	4.7%	20.83	14.41
99.5%	5.8%	16.67	11.52
99.4%	7.0%	13.89	9.60
99.3%	8.1%	11.90	8.22
99.2%	9.2%	10.42	7.19
99.1%	10.3%	9.26	6.39
99.0%	11.4%	8.33	5.75
98.9%	12.4%	7.58	5.22
98.8%	13.5%	6.94	4.79
98.7%	14.5%	6.41	4.41
98.6%	15.6%	5.95	4.10
98.5%	16.6%	5.56	3.82
98.4%	17.6%	5.21	3.58
98.3%	18.6%	4.90	3.37
98.2%	19.6%	4.63	3.18
98.1%	20.6%	4.39	3.01
98.0%	21.5%	4.17	2.86

A regime switching example

To show how regime switching is incorporated in the inflation generator, we walk through a sample path that illustrates a regime shift. The inflation generator uses three regimes and at any point in time, the inflation process follows an autoregressive model like equation (4) using monthly time steps. However, if there is a regime change in the economy, the parameters of the process change. Table 11 shows one example of parameters for the three regimes.

TABLE 11

Parameters for Regime Shift Simulation Example

Regime	k	θ	σ
Normal	1.0	3.5%	5.0%
High	0.4	9.0%	3.0%
Deflation	0.6	-3.0%	3.0%

Table 12 shows one sample path in a simulation, where the inflation process begins in the "normal" regime. In this example, after three months it turns out that the economy switches to a "high inflation" regime.

TABLE 12

Month by Month Projection for a Single Projection

Time Step	***Regime***	***Normal Draw***	***Simulated Annualized Inflation***
0	Normal		1.00%
1/12	Normal	-0.41144	0.61%
2/12	Normal	-0.13124	0.67%
3/12	Normal	1.29302	2.77%
4/12	High	-0.15564	1.48%
5/12	High	0.52153	2.18%

We simulate inflation for the next five months:

$$q_t + k(\theta - q_t)\Delta t + \varepsilon_t \sigma \sqrt{\Delta t} = q_{t+1}$$

$$\text{Month 1: } 1.00\% + 1.0 \times (3.50\% - 1.00\%) \times \frac{1}{12} - 0.41144 \times 5.0\% \times \sqrt{\frac{1}{12}} = 0.61\%$$

$$\text{Month 2: } 0.61\% + 1.0 \times (3.50\% - 0.61\%) \times \frac{1}{12} - 0.13124 \times 5.0\% \times \sqrt{\frac{1}{12}} = 0.61\%$$

$$\text{Month 3: } 0.67\% + 1.0 \times (3.50\% - 0.67\%) \times \frac{1}{12} + 1.29302 \times 5.0\% \times \sqrt{\frac{1}{12}} = 2.77\%$$

After the third month, we switch regimes and therefore, the parameters of the autoregressive inflation process change.

$$\text{Month 4: } 2.77\% + 0.4 \times (9.00\% - 2.77\%) \times \frac{1}{12} - 0.15564 \times 3.0\% \times \sqrt{\frac{1}{12}} = 1.48\%$$

$$\text{Month 5: } 1.48\% + 0.4 \times (9.00\% - 1.48\%) \times \frac{1}{12} + 0.52153 \times 3.0\% \times \sqrt{\frac{1}{12}} = 2.18\%$$

4. USING THE INFLATION GENERATOR

This section walks the user through the inflation generator spreadsheet. To run the inflation generator, the user must complete the following steps:

1. Open the inflation model spreadsheet and enable the macros.
2. Review and adjust the model's parameters on the "ModelInput" sheet (if desired).
3. Select the cells of interest to track during the simulation (select output cells).
4. Run the simulation.
5. Review the output.

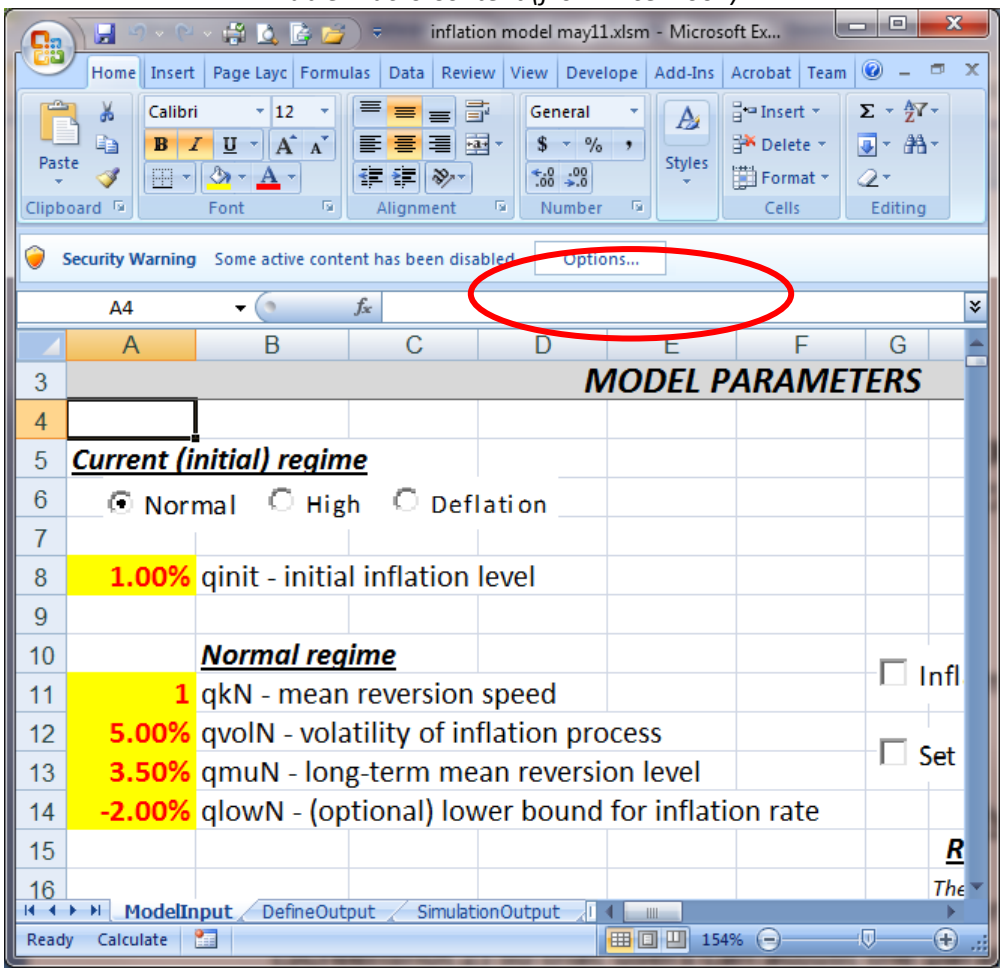
The following documentation walks the user through these steps.

Loading the inflation model

After opening the spreadsheet you will have to enable the embedded macros. Depending on your version of Excel, you may get a popup window or a security warning near the top of the screen (shown as Screenshot #1 below) asking to enable the macro content.

SCREENSHOT 1

Enable Macro Content (from Excel 2007)



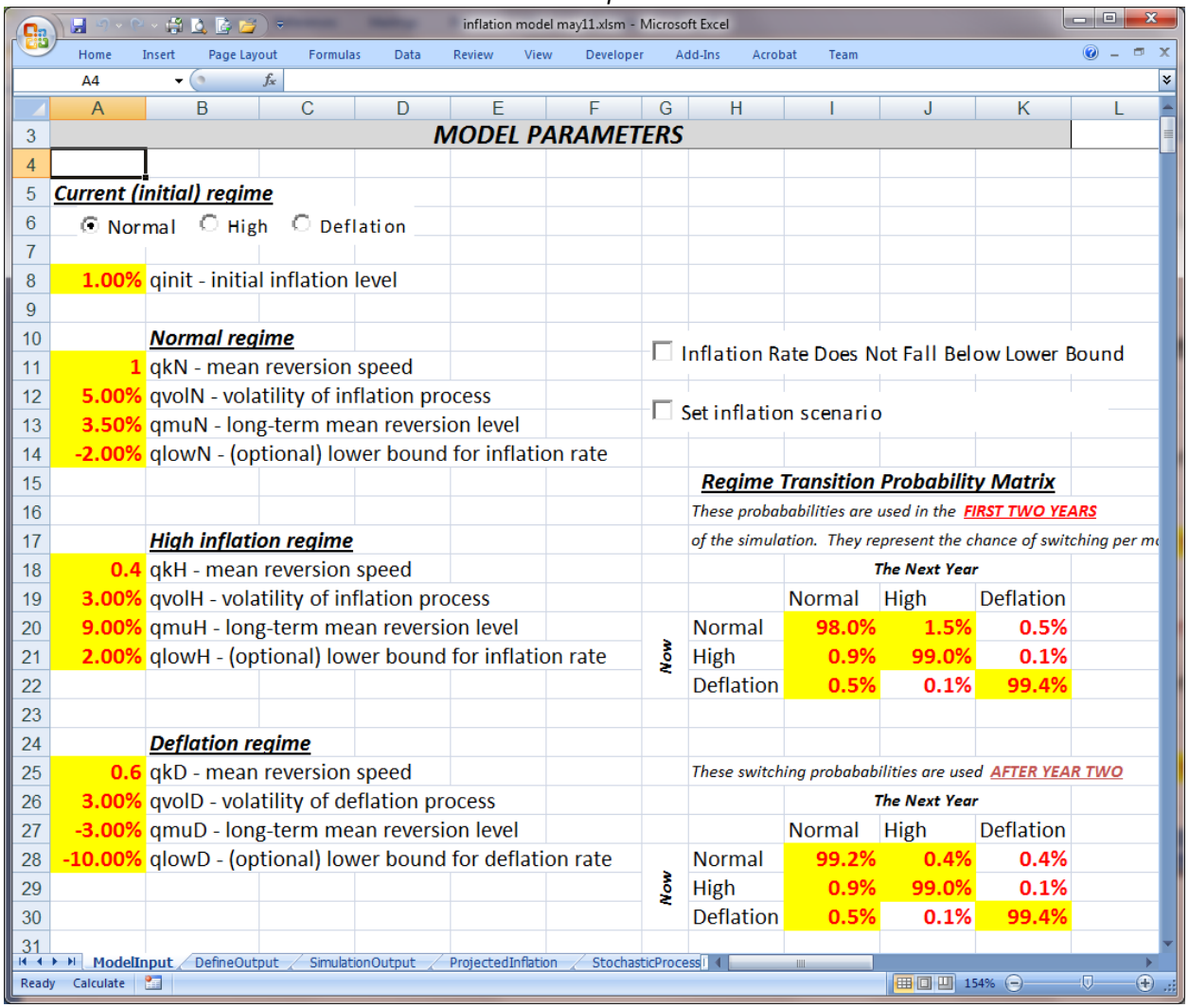
After the macros have been enabled, the user must accept the legal disclaimer before continuing to the model input sheet.

Adjusting the parameters - The "ModelInput" sheet

This section of the documentation walks you through each of the inputs on the "ModelInput" sheet (Screenshot 2) so that users can adjust the parameters of the model, if desired.

SCREENSHOT 2

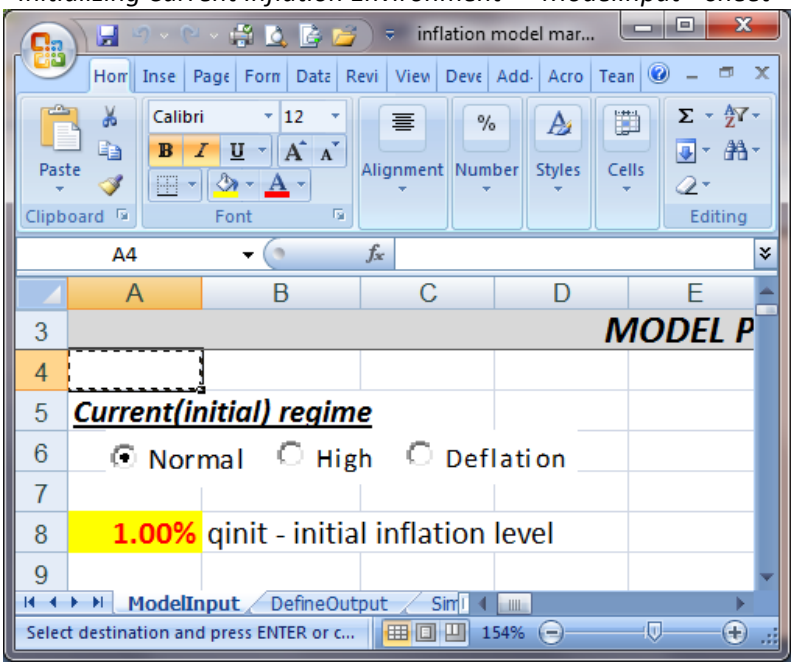
The "ModelInput" Sheet


inflation model may11.xlsm - Microsoft Excel
MODEL PARAMETERS
Current (initial) regime
Normal
High
Deflation
1.00% qinit - initial inflation level
Normal regime
1 qkN - mean reversion speed
5.00% qvolN - volatility of inflation process
3.50% qmuN - long-term mean reversion level
-2.00% qlowN - (optional) lower bound for inflation rate
High inflation regime
0.4 qkH - mean reversion speed
3.00% qvolH - volatility of inflation process
9.00% qmuH - long-term mean reversion level
2.00% qlowH - (optional) lower bound for inflation rate
Deflation regime
0.6 qkD - mean reversion speed
3.00% qvolD - volatility of deflation process
-3.00% qmuD - long-term mean reversion level
-10.00% qlowD - (optional) lower bound for deflation rate
Inflation Rate Does Not Fall Below Lower Bound
Set inflation scenario
Regime Transition Probability Matrix
These probabilities are used in the FIRST TWO YEARS
The Next Year
Normal High Deflation
Now
Normal 98.0% 1.5% 0.5%
High 0.9% 99.0% 0.1%
Deflation 0.5% 0.1% 99.4%
These switching probabilities are used AFTER YEAR TWO
The Next Year
Normal High Deflation
Now
Normal 99.2% 0.4% 0.4%
High 0.9% 99.0% 0.1%
Deflation 0.5% 0.1% 99.4%
ModelInput
DefineOutput
SimulationOutput
ProjectedInflation
StochasticProcess
Ready
Calculate
154%

The top left of the screen (Screenshot 3) asks the user to initialize the current inflation environment. The default regime is checked as "Normal", but the user can click on any of the other radio buttons to begin simulations in an alternate regime. In addition, the most recent value for the level of inflation should be entered (labeled "qinit" in Screenshot3).

SCREENSHOT 3

Initializing Current Inflation Environment – "ModelInput" sheet

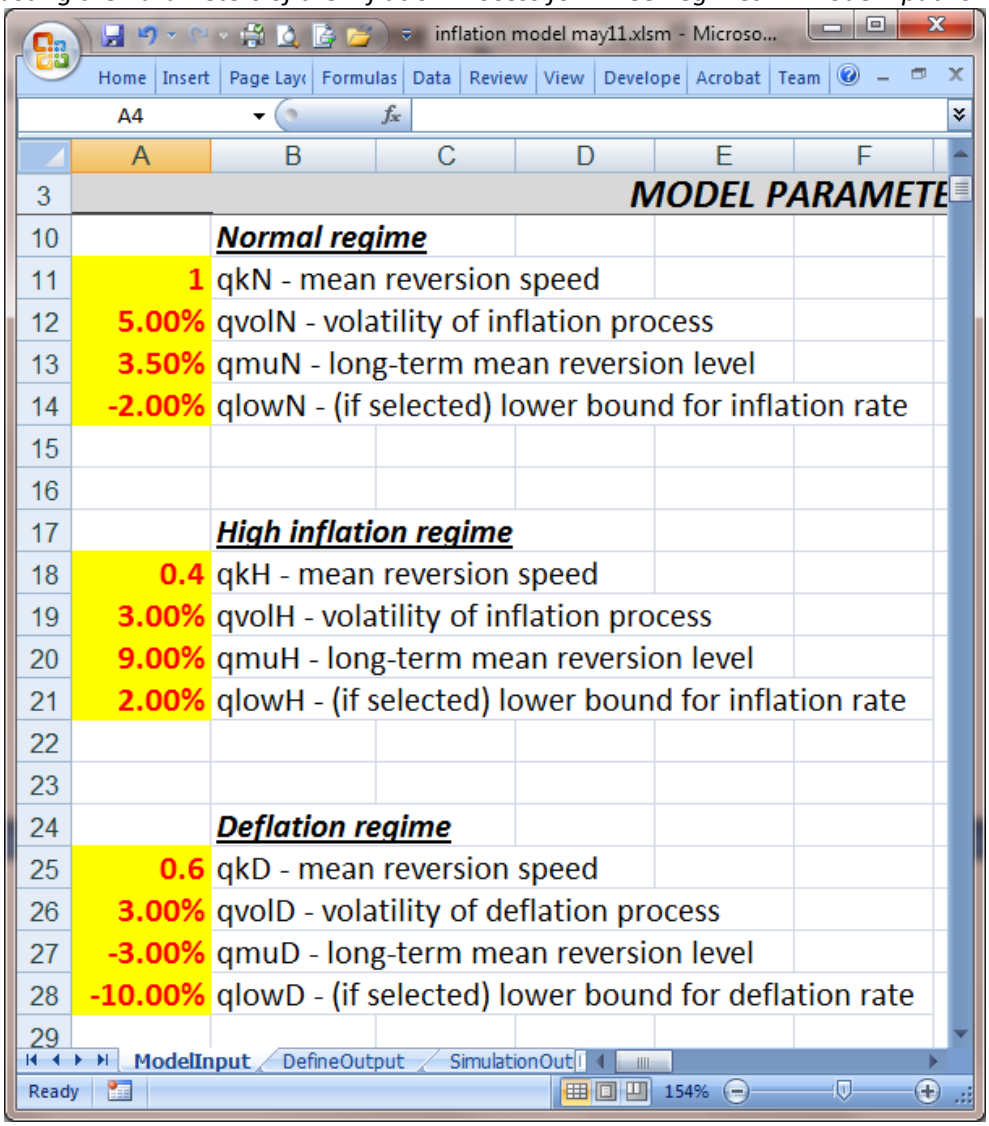


As discussed above, the inflation generator is based on a regime switching process with three regimes. One interpretation of these regimes is that when the US economy is experiencing "Normal" economic times, the average inflation rate is considered moderate. But two other economic regimes are possible in the future. First, expansionary fiscal policy combined with accommodative monetary policy may lead to sustained inflationary pressures. In this "High" inflation regime, there is a significantly higher average level of inflation than indicated from recent history. It seems plausible that in this high inflation regime, volatility may also be higher. A second alternate regime is that of continued worldwide economic stagnation despite government spending and central bank easing. The third regime incorporated in the inflation model reflects the possibility of these deflationary pressures. In this regime, the average level of inflation and the volatility are low.

Screenshot 4 shows the parameters for each of these regimes. See the discussion in the first section of this paper to see the effects of changes in these parameters on inflation projections.

SCREENSHOT 4

Adjusting the Parameters of the Inflation Process for Three Regimes – "ModelInput" sheet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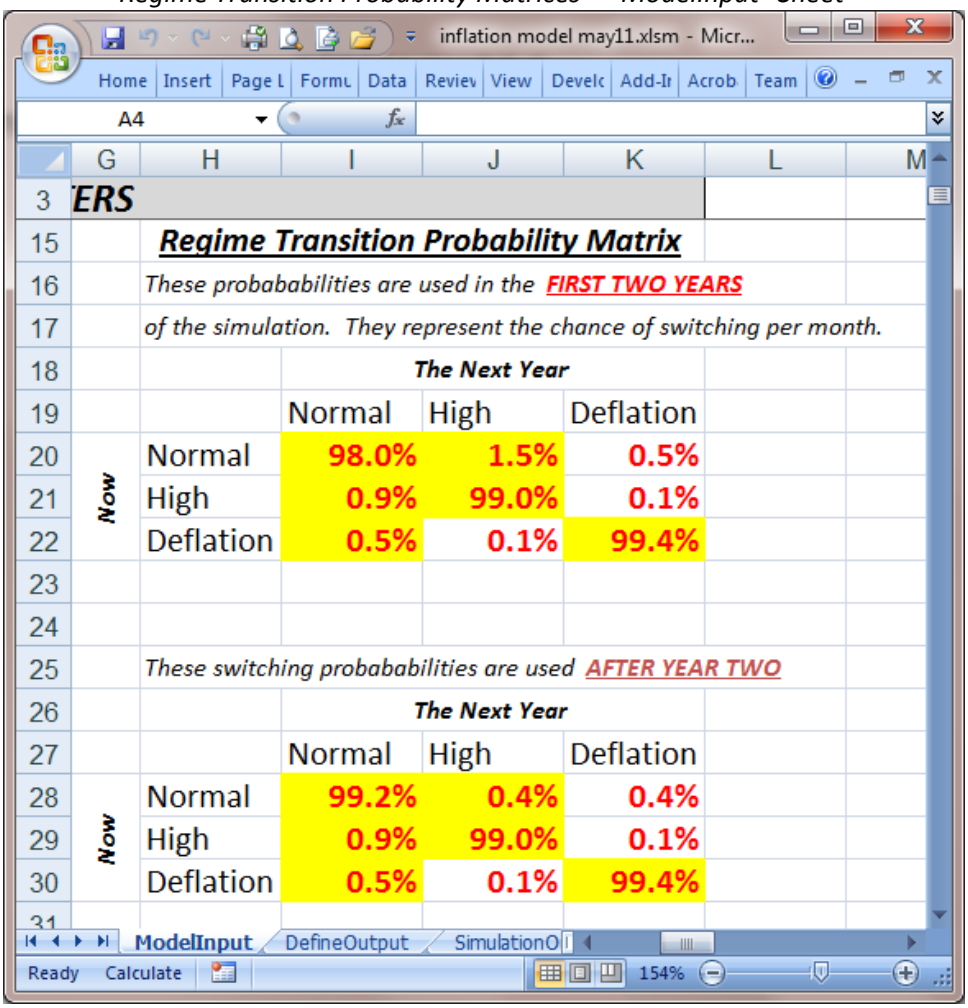
MODEL PARAMETE
Normal regime
1 qkN - mean reversion speed
5.00% qvolN - volatility of inflation process
3.50% qmuN - long-term mean reversion level
-2.00% qlowN - (if selected) lower bound for inflation rate
High inflation regime
0.4 qkH - mean reversion speed
3.00% qvolH - volatility of inflation process
9.00% qmuH - long-term mean reversion level
2.00% qlowH - (if selected) lower bound for inflation rate
Deflation regime
0.6 qkD - mean reversion speed
3.00% qvolD - volatility of deflation process
-3.00% qmuD - long-term mean reversion level
-10.00% qlowD - (if selected) lower bound for deflation rate
ModelInput DefineOutput SimulationOut
Ready
154%

The transition among the various regimes is directed by two transition probability matrices and these are shown in Screenshot 5. The first matrix is used for the first two years of the simulation while the second transition probability matrix is used after year two. Having separate matrices allows the users to tilt short-term probabilities toward specific regimes that the user may feel are warranted because of existing economic conditions. For example, to increase the probabilities over the next two years to the high inflation regime, adjust the short-term probability matrix to reflect this. After two years, the regime switching probabilities of the second matrix are applied.

SCREENSHOT 5

Regime Transition Probability Matrices – "ModelInput" Sheet



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A4

Regime Transition Probability Matrix

These probabilities are used in the ***FIRST TWO YEARS***

of the simulation. They represent the chance of switching per month.

Now	The Next Year: Normal	High	Deflation
Normal	98.0%	1.5%	0.5%
High	0.9%	99.0%	0.1%
Deflation	0.5%	0.1%	99.4%

These switching probabilities are used ***AFTER YEAR TWO***

Now	The Next Year: Normal	High	Deflation
Normal	99.2%	0.4%	0.4%
High	0.9%	99.0%	0.1%
Deflation	0.5%	0.1%	99.4%

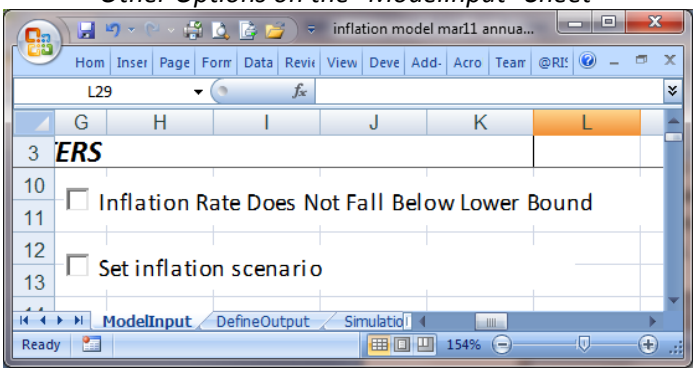
ModelInput | DefineOutput | SimulationO

Ready | Calculate | 154%

Two other options are available on the model input sheet (Screenshot 6). These options allow the user to adjust the inflation rate resulting from the simulations. The first option, when checked, does not allow the inflation process to fall below a user defined lower bound. For example, a specific user may decide to disallow negative inflation rates (deflationary scenarios) during a normal economic environment. To use a lower bound, the user would check the lower bound box and enter the desired lower bound as an additional parameter of the inflation process (see the parameters labeled "qlowN", "qlowH", and "qlowD" in Screenshot 4). When the lower bound option is checked, if a simulated interest rate falls below the user defined lower bound, the spreadsheet replaces the simulated inflation rate with the desired lower bound.

SCREENSHOT 6

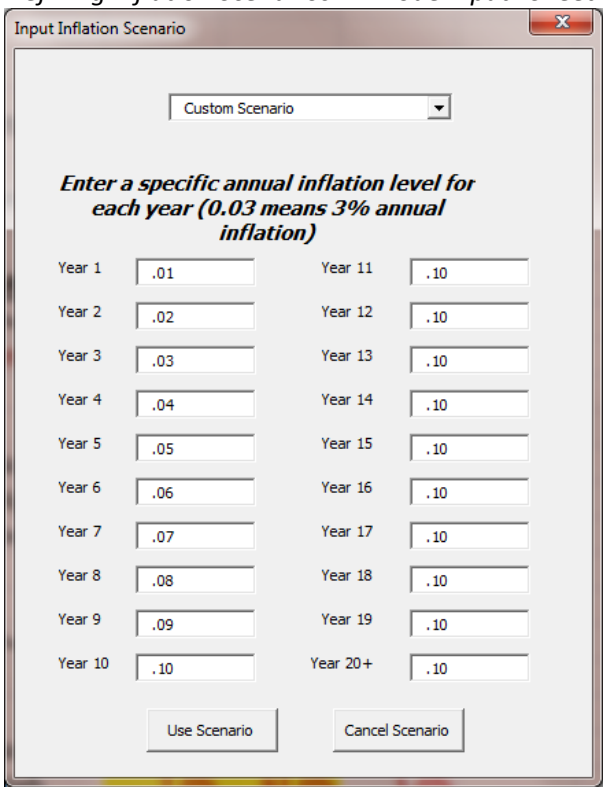
Other Options on the "ModelInput" Sheet



The second checkbox option shown in Screenshot 6 may be useful if the inflation generator is used as an engine for other applications. When the second box is checked to "Set inflation scenario", all stochastic processes are overridden by a user defined inflation scenario. This allows the user to test specific path of inflation that is of concern for a particular application. When checking the box, a pop-up window (Screenshot 7) asks the user to define the specific inflation path. After entering the path, if the user would like to stop using defined scenarios, simply unselect the "Set inflation scenario" checkbox.

SCREENSHOT 7

Defining Inflation Scenarios – "ModelInput" Sheet

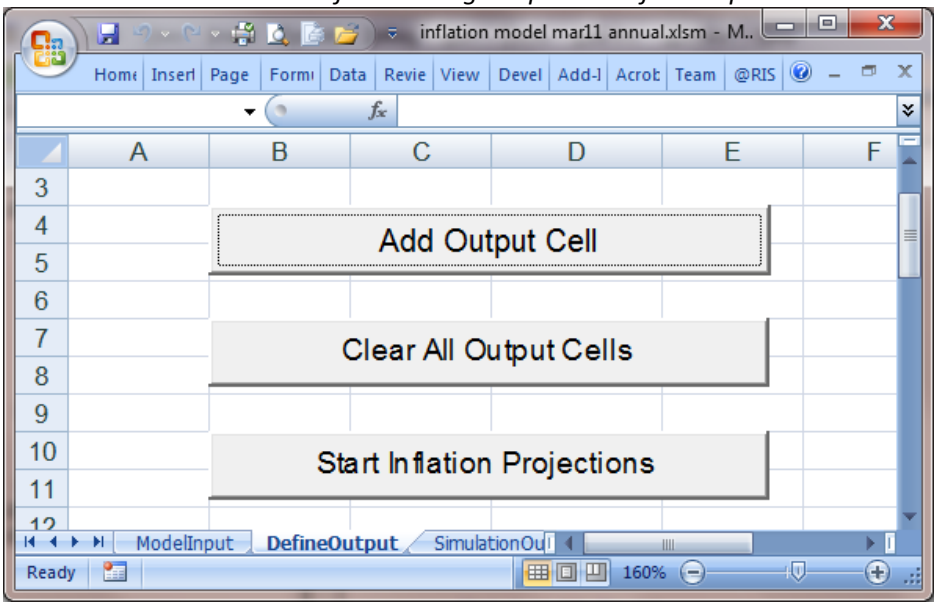


Defining Output Cells

The sheet labeled "DefineOutput" contains three command buttons that execute macros to help users track specific cells of interest, start a projection, and review simulation results (Screenshot 8).

SCREENSHOT 8

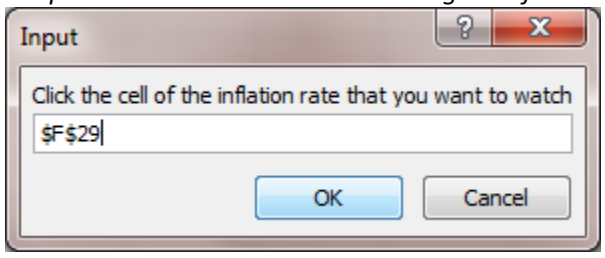
Three Macro Buttons for Tracking Output – "DefineOutput" Sheet



Before running a projection, the user will want to define the output cells of interest to track in the simulation. When clicking the "Add Output Cell" button, a pop-up window (Screenshot 9) takes you to the model's inflation rate output and the user chooses a particular cell to follow.

SCREENSHOT 9

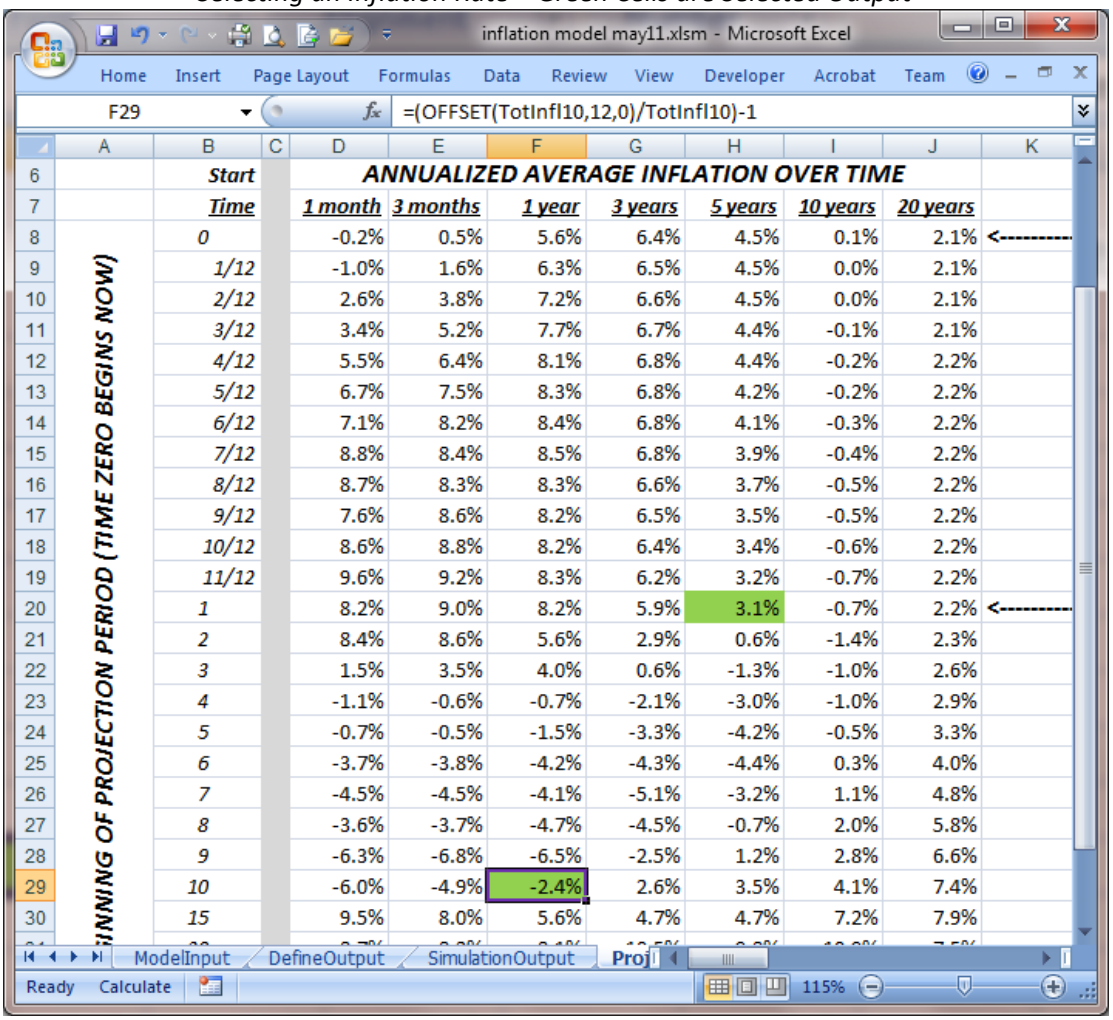
"Add Output Cell" Macro Button – Selecting an Inflation Rate



Note that in the example, the user is adding the one-year inflation rate that starts in ten years (see cell F29 in Screenshot 10). Users can choose up to twenty variables to follow and after choosing a cell, these variables of interest are shaded green.

SCREENSHOT 10

Selecting an Inflation Rate – Green Cells are Selected Output


inflation model may11.xlsm - Microsoft Excel
Home Insert Page Layout Formulas Data Review View Developer Acrobat Team
F29
=(OFFSET(TotInfl10,12,0)/TotInfl10)-1
ANNUALIZED AVERAGE INFLATION OVER TIME
BEGINNING OF PROJECTION PERIOD (TIME ZERO BEGINS NOW)
ModelInput DefineOutput SimulationOutput Proj
Ready Calculate
115%

	Start Time	1 month	3 months	1 year	3 years	5 years	10 years	20 years	
8	0	-0.2%	0.5%	5.6%	6.4%	4.5%	0.1%	2.1%	<----------
9	1/12	-1.0%	1.6%	6.3%	6.5%	4.5%	0.0%	2.1%	
10	2/12	2.6%	3.8%	7.2%	6.6%	4.5%	0.0%	2.1%	
11	3/12	3.4%	5.2%	7.7%	6.7%	4.4%	-0.1%	2.1%	
12	4/12	5.5%	6.4%	8.1%	6.8%	4.4%	-0.2%	2.2%	
13	5/12	6.7%	7.5%	8.3%	6.8%	4.2%	-0.2%	2.2%	
14	6/12	7.1%	8.2%	8.4%	6.8%	4.1%	-0.3%	2.2%	
15	7/12	8.8%	8.4%	8.5%	6.8%	3.9%	-0.4%	2.2%	
16	8/12	8.7%	8.3%	8.3%	6.6%	3.7%	-0.5%	2.2%	
17	9/12	7.6%	8.6%	8.2%	6.5%	3.5%	-0.5%	2.2%	
18	10/12	8.6%	8.8%	8.2%	6.4%	3.4%	-0.6%	2.2%	
19	11/12	9.6%	9.2%	8.3%	6.2%	3.2%	-0.7%	2.2%	
20	1	8.2%	9.0%	8.2%	5.9%	3.1%	-0.7%	2.2%	<----------
21	2	8.4%	8.6%	5.6%	2.9%	0.6%	-1.4%	2.3%	
22	3	1.5%	3.5%	4.0%	0.6%	-1.3%	-1.0%	2.6%	
23	4	-1.1%	-0.6%	-0.7%	-2.1%	-3.0%	-1.0%	2.9%	
24	5	-0.7%	-0.5%	-1.5%	-3.3%	-4.2%	-0.5%	3.3%	
25	6	-3.7%	-3.8%	-4.2%	-4.3%	-4.4%	0.3%	4.0%	
26	7	-4.5%	-4.5%	-4.1%	-5.1%	-3.2%	1.1%	4.8%	
27	8	-3.6%	-3.7%	-4.7%	-4.5%	-0.7%	2.0%	5.8%	
28	9	-6.3%	-6.8%	-6.5%	-2.5%	1.2%	2.8%	6.6%	
29	10	-6.0%	-4.9%	-2.4%	2.6%	3.5%	4.1%	7.4%	
30	15	9.5%	8.0%	5.6%	4.7%	4.7%	7.2%	7.9%	

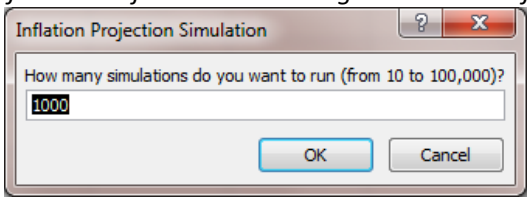
If the user would like to clear all variables previously selected to start with a new batch of output variables, click on button labeled “Clear All Output Cells” on the “DefineOutput” sheet (Screenshot 8).

Running the simulation

The last button on the "DefineOutput" sheet (Screenshot 8) is to "Start Inflation Projections," which projects a number of inflation paths over the next 50 years based on the number of simulations entered by the user.

SCREENSHOT 11

Start Inflation Projections – Entering the Number of Paths

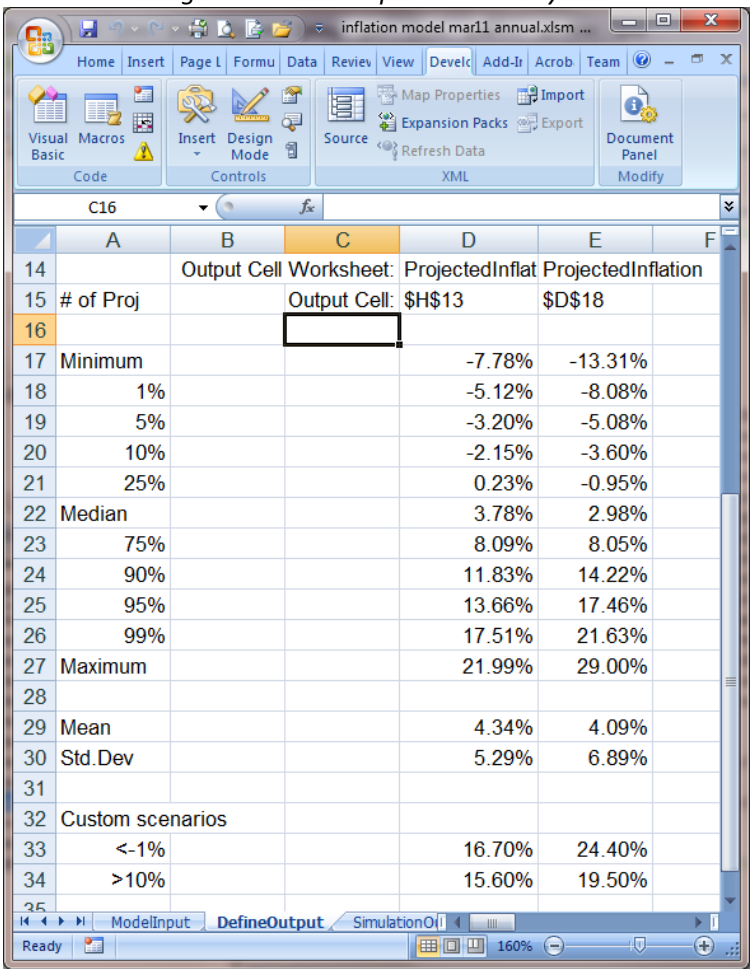


Reviewing the simulation output

After the model has simulated the desired number of inflation paths, the Define Output worksheet will show the summary statistics from the simulation including mean, standard deviation, percentiles of the distribution, and a few (adjustable) scenarios.

SCREENSHOT 12

Reviewing Simulation Output – Summary Statistics

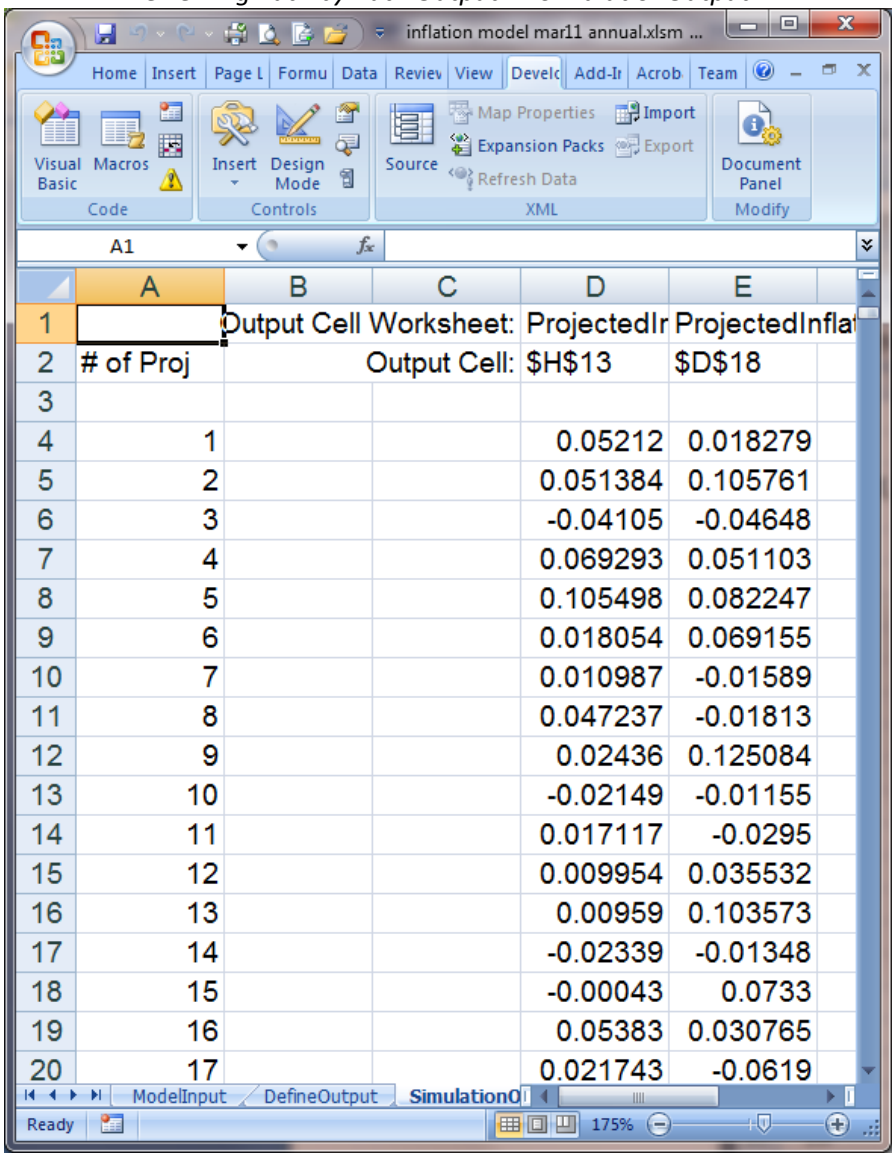


	A	B	C	D	E
14		Output Cell	Worksheet:	ProjectedInflat	ProjectedInflation
15	# of Proj		Output Cell:	H13	D18
16					
17	Minimum			-7.78%	-13.31%
18	1%			-5.12%	-8.08%
19	5%			-3.20%	-5.08%
20	10%			-2.15%	-3.60%
21	25%			0.23%	-0.95%
22	Median			3.78%	2.98%
23	75%			8.09%	8.05%
24	90%			11.83%	14.22%
25	95%			13.66%	17.46%
26	99%			17.51%	21.63%
27	Maximum			21.99%	29.00%
28					
29	Mean			4.34%	4.09%
30	Std.Dev			5.29%	6.89%
31					
32	Custom scenarios				
33	<-1%			16.70%	24.40%
34	>10%			15.60%	19.50%

Instead of just the summary statistics, if the user wishes to review all of the output (for each and every simulated path), these can be seen on the "SimulationOutput" sheet (Screenshot 13).

SCREENSHOT 13

Reviewing Path by Path Output – "SimulationOutput"

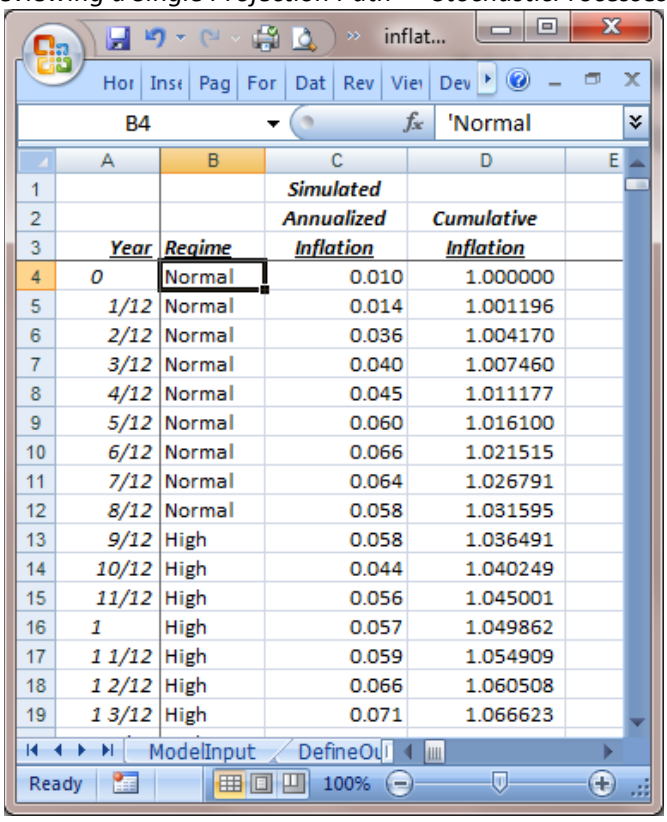


	A	B	C	D	E
1		Output Cell Worksheet:		ProjectedIr	ProjectedInfla
2	# of Proj		Output Cell:	H13	D18
3					
4	1			0.05212	0.018279
5	2			0.051384	0.105761
6	3			-0.04105	-0.04648
7	4			0.069293	0.051103
8	5			0.105498	0.082247
9	6			0.018054	0.069155
10	7			0.010987	-0.01589
11	8			0.047237	-0.01813
12	9			0.02436	0.125084
13	10			-0.02149	-0.01155
14	11			0.017117	-0.0295
15	12			0.009954	0.035532
16	13			0.00959	0.103573
17	14			-0.02339	-0.01348
18	15			-0.00043	0.0733
19	16			0.05383	0.030765
20	17			0.021743	-0.0619

Finally, the "StochasticProcess" sheet shows a bit of the internal operations of the model (Screenshot 14). It illustrates *one path* of inflation including the simulated regime for each year. If the user would like to simulate a single path at a time to see the model project, simply recalculate the spreadsheet by hitting F9.

SCREENSHOT 14

Reviewing a Single Projection Path – "StochasticProcesses"



	A	B	C	D
1			*Simulated*	
2			*Annualized*	*Cumulative*
3	*Year*	*Regime*	*Inflation*	*Inflation*
4	*0*	Normal	0.010	1.000000
5	*1/12*	Normal	0.014	1.001196
6	*2/12*	Normal	0.036	1.004170
7	*3/12*	Normal	0.040	1.007460
8	*4/12*	Normal	0.045	1.011177
9	*5/12*	Normal	0.060	1.016100
10	*6/12*	Normal	0.066	1.021515
11	*7/12*	Normal	0.064	1.026791
12	*8/12*	Normal	0.058	1.031595
13	*9/12*	High	0.058	1.036491
14	*10/12*	High	0.044	1.040249
15	*11/12*	High	0.056	1.045001
16	*1*	High	0.057	1.049862
17	*1 1/12*	High	0.059	1.054909
18	*1 2/12*	High	0.066	1.060508
19	*1 3/12*	High	0.071	1.066623

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